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2006 J. Phys. A: Math. Gen. 39 L673

(http://iopscience.iop.org/0305-4470/39/50/L01)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **39** (2006) L673–L675 doi:10.1088/0305-4470/39/50/L01

**LETTER TO THE EDITOR**

# $\mathbf{PSL}_n(q)$ as operator group of isospectral drums

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Received 7 September 2006, in final form 9 November 2006
Published 30 November 2006
Online at stacks.iop.org/JPhysA/39/L673

**Abstract**

In a paper by M Kac (1966 *Am. Math. Mon.* **73** 1–23), Kac asked his famous question 'Can one hear the shape of a drum?', which was answered negatively in Gordon *et al* (1992 *Invent. Math.* **110** 1–22) by construction of planar isospectral pairs. In Buser *et al* (1994 *Int. Math. Res. Not.* **9**), it is observed that all operator groups associated with the known counter examples are isomorphic to one of $\mathbf{PSL}_3(2)$, $\mathbf{PSL}_3(3)$, $\mathbf{PSL}_4(2)$ and $\mathbf{PSL}_3(4)$. We show that if $(D_1, D_2)$ is a pair of non-congruent planar isospectral domains constructed from unfolding a polygonal base-tile and with associated operator group $\mathbf{PSL}_n(q)$, then $(n, q)$ belongs to this very restricted list.

PACS numbers: 02.70.Hm, 03.65.Nk, 03.65.Ge, 05.45.−a

## 1. Introduction

A celebrated question of M Kac [7] asks whether simply connected domains in $\mathbb{R}^2$ for which the sets $\{\lambda_n \parallel n \in \mathbb{N}\}$ of solutions (eigenvalues) of the stationary Schrödinger equation

$$(\Delta + \lambda)\Psi = 0 \qquad \text{with} \qquad \Psi_{|\text{Boundary}} = 0$$

coincide, are necessarily congruent.

Counter examples were constructed to the analogous question on Riemannian manifolds (cf R Brooks [1]), but for Euclidian domains the question appeared to be much harder. C Gordon, D Webb and S Wolpert constructed a pair of simply connected non-isometric Euclidian isospectral domains—also called 'planar isospectral pairs' or 'isospectral billiards'—in [5]. Other examples were found later; see for instance the paper by P Buser, J Conway, P Doyle and K-D Semmler [2], which contains all known planar examples constructed from unfolding a base-tile with three sides. There, in table 1 of *op. cit.*, it is shown that

> the operator groups of these known examples are all isomorphic to the classical group $\mathbf{PSL}_n(q)$, where $(n, q) \in \{(3, 2), (3, 3), (4, 2), (3, 4)\}$.

In this letter, we show, rather unexpectedly, that

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if $(D_1, D_2)$ is a pair of non-congruent planar isospectral domains constructed from unfolding any polygonal base-tile and with associated operator group $\mathbf{PSL}_n(q)$, $n \geqslant 2$, then $(n, q)$ belongs to the very restricted list $\{(3, 2), (3, 3), (4, 2)$ and $(3, 4)\}$ (main result).

For more details on the theory surrounding Kac's question, see [4].

In the papers [3, 8, 9], the authors studied an abstraction of certain properties of sets of involutions of projective spaces over finite fields, which arise naturally in the construction theory of isospectral billiards—see the following section for formal details.

The essential idea of the present letter is that if we assume that the associated operator groups are projective special linear groups in *any dimension* $n$, $n \geqslant 2$, the situation which we want to consider can be reduced to those handled in [3, 8, 9].

In [3, 8, 9], the reason for studying these sets of involutions is that there could arise certain data which yield new counter examples to Kac's question by the tiling method.

Here, our viewpoint is entirely different; we want to show, using [3, 8, 9], that in fact only very few counter examples by the tiling method arise when the operator group is $\mathbf{PSL}_n(q)$, $n \geqslant 2$.

## 2. Proof of the main result

For a pair of isospectral billiards on $N$ copies of a tile with $r$ sides, one needs $r$ involutions acting on a set of $N$ letters, with the property that the graph $\Gamma = \Gamma(V, E)$, where the vertex set $V$ is the set of $N$ letters (tiles), and two vertices are joined by an edge of $E$ if the corresponding letters are interchanged by at least one of the involutions, has no closed circuits and is connected. For any involution the number of edges is $(N - s)/2$, with $s$ the number of fixed points of the involution. The total number of edges must equal $N - 1$, and the group of transformations generated by the involutions must act transitively on the set of $N$ points. We will call this group the 'operator group' of the billiard.

Now suppose that $(D_1, D_2)$ is a pair of non-congruent planar isospectral domains constructed from unfolding an $\ell$-gon, $\ell \geqslant 3$, $N < \infty$ times as just described. Since the $D_i$ are constructed by unfolding an $\ell$-gon, we can associate $\ell$ involutions $\theta_i^{(j)}$ to $D_i$, $j = 1, 2, \ldots, \ell$ and $i = 1, 2$. Define the operator groups

$$G_i = \langle \theta_i^{(j)} \rangle.$$

Now suppose that

$$G_1 \cong \mathbf{PSL}_n(q) \cong G_2,$$

with $q$ a prime power and $n \geqslant 2$ a natural number.

The natural geometry on which $\mathbf{PSL}_n(q)$ acts (faithfully) is the $(n - 1)$-dimensional projective space $\mathbf{PG}(n - 1, q)$ over the finite field $\mathbf{GF}(q)$ [6]. It should be mentioned that $\mathbf{PSL}_n(q)$ acts transitively on the points of $\mathbf{PG}(n - 1, q)$. So we can see the involutions $\theta_i^{(j)}$ for fixed $i \in \{1, 2\}$ as automorphisms of $\mathbf{PG}(n - 1, q)$ that generate $\mathbf{PSL}_n(q)$.

This means that for fixed $i \in \{1, 2\}$ the triple

$$\left(\mathbf{PG}(n - 1, q), \left\{\theta_i^{(j)}\right\}, \ell\right)$$

yields 'generalized projective isospectral data' in the sense of [9][2].

[2] This is a triple $(\mathbf{P}, \{\theta^{(i)}\}, r)$, where $\mathbf{P}$ is a finite projective space of dimension at least 2, and $\{\theta^{(i)}\}$ a set of $r$ nontrivial involutory automorphisms of $\mathbf{P}$, satisfying

$$r(|\mathbf{P}|) - \sum_{j=1}^{r} \mathrm{Fix}(\theta^{(j)}) = 2(|\mathbf{P}| - 1), \tag{1}$$

for some natural number $r \geqslant 3$.

These data were completely classified in [9], mentioning that in [3] the planar case was handled for involutions with the same number of fixed points, and in [8] the general case for involutions fixing the same number of points.

**Theorem 2.1** ([9]). *Let* $\mathbf{P} = \mathbf{PG}(l, q)$ *be the* $l$*-dimensional projective space over the finite field* $\mathbf{GF}(q)$, $l \geqslant 2$, *and suppose there exist generalized projective isospectral data* $(\mathbf{P}, \{\theta^{(i)}\}, r)$ *which yield isospectral billiards. Then either* $l = 2$, *the* $\theta^{(i)}$ *fix the same number of points of* $\mathbf{P}$, *and the solutions are as described in [3], or* $l = 3, r = 3$ *and* $q = 2$, *and again the examples can be found in [3].*

This theorem implies that $(n, q)$ is contained in $\{(3, 2), (3, 3), (4, 2), (3, 4)\}$ if $n \geqslant 3$.

Now suppose that $n = 2$. We have to solve the equation

$$\ell|\mathbf{PG}(1, q)| - \sum_{j=1}^{\ell} \mathrm{Fix}\big(\theta_i^{(j)}\big) = 2(|\mathbf{PG}(1, q)| - 1),$$

for fixed $i \in \{1, 2\}$, where $\mathrm{Fix}\big(\theta_i^{(j)}\big)$ is the number of fixed points in $\mathbf{PG}(1, q)$ of $\theta_i^{(j)}$. Since $|\mathbf{PG}(1, q)| = q + 1$ and since a nontrivial element of $\mathbf{PSL}_2(q)$ fixes at most two points of $\mathbf{PG}(1, q)$, an easy calculation leads to a contradiction if $q \geqslant 3$.

Now let $q = 2$. Then $\mathbf{PSL}_2(2)$ contains precisely three involutions, and they each fix precisely one point of $\mathbf{PG}(1, 2)$. A numerical contradiction follows.

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